The Quotient l-Infinity Norm: Applications and Relation to Hilbert's Projective Metric

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Abstract

This paper explores multiple applications of the quotient l^∞ norm, and how it relates to Hilbert's projective metric. We introduce a simple characterization of Hilbert's projective metric using the quotient l^∞ norm applied to log-transformed vectors, as well as a similar characterization of Thompson's metric using the ordinary l^∞ norm. Using our formulations of the two metrics, we are able to prove relevant homeomorphisms which allow for easier characterization of topological properties of sets under Hilbert's projective metric or Thompson's metric. This allows for a simplification of the proof of the Perron-Frobenius theorem as given in Kohlberg (1982). We study the geometry of the quotient l^∞ unit ball, and prove that in n dimensions, the unit ball is a polytope with $2^n - 2$ extreme points. This allows us to study the geometry of the unit ball in higher dimensions.

Hilbert's Projective Metric & The Quotient l^∞ Norm

In this section, we will demonstrate that Hilbert's projective metric is closely related to a norm-induced metric we can construct using the quotient l^∞ norm. Before we do so, we introduce a few definitions.

Definition 1. *(Thompson, 1963) Let V be a real, normed vector space and let C be a nonempty, closed subset of V. Then we say C is a positive convex cone if it satisfies the following conditions:*

- $x \in C, \alpha > 0 \implies \alpha x \in C$ *(cone condition)*
- $x, y \in C \implies x + y \in C$ *(convexity condition)*
- $x \in C, -x \in C \implies x = 0$ *(positivity condition)*

Definition 2. *(Mathematical Society of Japan and Itô, 1993) Let F be an ordered field. F is said to be an Archimedean ordered field if for all positive $x, y \in F$, there exists positive $n \in \mathbb{N}$ such that $nx > y$.*

We can define an equivalence relation on a cone C as follows: for $x, y \in C$, $x \equiv y$ if $x = ky$ for some $k \in \mathbb{R}_+$. With this equivalence relation, the equivalence classes $[x]$ are called the "rays" of our cone C. Hilbert's projective metric which we define below is a metric that acts on the rays of a cone. The following definition of Hilbert's projective metric is based on those given in Bushell (1973) and Koufany (2004).

Let V be a real Banach space and let $C \subseteq V$ be a positive convex cone whose interior Ω is nonempty. We define a relation $(V, \leq_C)$ as follows: $x \leq_C y$ if and only if $y - x \in C$. We require the partial order $(V, \leq_C)$ to be Archimedean. For $x, y \in \Omega$, we let

$$M(x, y) := \inf\{\lambda : x \leq_C \lambda y\},$$

and

$$m(x, y) := \sup\{\mu : \mu y \leq_C x\}.$$

Note that the Archimedean condition ensures both M and m are finite quantities. We can now define Hilbert's projective metric on Ω by

$$d_{HPM}([x], [y]) = \log \frac{M(x, y)}{m(x, y)}.$$

If our cone is $\mathbb{R}^n_+$, Hilbert's projective metric can be given by

$$d_{HPM}([x], [y]) = \log \frac{\max \frac{x_i}{y_i}}{\min \frac{x_j}{y_j}} = \log \max_{i,j} \frac{x_i y_j}{y_i x_j}.$$

We will only be considering Hilbert's projective metric on the positive cone $P = \mathbb{R}^n_+$. On P, we define the same equivalence relation that we introduced earlier: $x \equiv y$ if $x = ky$ for some $k \in \mathbb{R}_+$. We denote the set of all equivalence classes of this space as $P/\sim$. Note that $P/\sim$ is the set of all positive rays in $\mathbb{R}^n$, and $(P/\sim, d_{HPM})$ is a metric space.

Next, we introduce quotient spaces and the quotient l^∞ norm. Let V be a vector space, and let S be a subspace of V. We then define an equivalence relation on elements of V as follows: we say $x \equiv y$ if $x - y \in S$. Then, the quotient space V/S is the set of all equivalence classes of V.

$$V/S = \{[x] : x \in V\}$$

V/S is a normed vector space with the norm

$$\|[x]\|_{V/S} = \inf_{s \in S} \|x - s\|.$$

In this paper, we will let $V = \mathbb{R}^n$ with the l^∞ norm. Further, let our subspace S be the span of the "all ones" vector given by $e = (1, 1, ..., 1)$. The quotient l^∞ norm on the space $\mathbb{R}^n/S$ is then given by

$$\|[x]\|_{\mathbb{R}^n/S} = \inf_t \|x - te\|_\infty.$$

We will show that Hilbert's projective metric and the quotient l^∞ norm as defined above are closely related. Consider a mapping from $P/\sim$ to $\mathbb{R}^n/S$ which takes $[(x_1, x_2, ..., x_n)] \in P/\sim$ to $[(\log(x_1), \log(x_2), ..., \log(x_n))] \in \mathbb{R}^n/S$. If we take the equivalence classes $[x] \in P/\sim$ and apply the log mapping we just defined, that gives us a mapping from the positive rays in $P/\sim$ to the quotient space $\mathbb{R}^n/S$. To demonstrate this, consider two equivalent vectors x and y in P. If $(x_1, ..., x_n) \equiv (y_1, ..., y_n)$ in the positive cone P, then it must be that $y_i = kx_i$ for some $k > 0$ and for all i. Taking the log map of x and y and subtracting them, we have:

$$\begin{aligned}
&(\log(x_1), ..., \log(x_n)) - (\log(y_1), ..., \log(y_n)) \\
&= (\log(x_1), ..., \log(x_n)) - (\log(kx_1), ..., \log(kx_n)) \\
&= (\log(x_1), ..., \log(x_n)) - (\log(x_1), ..., \log(x_n)) - \log(k)e \\
&= -log(k)e,
\end{aligned}$$

where $e = (1, 1, ..., 1)$. We see that $(\log(x_1), ..., \log(x_n)) - (\log(y_1), ..., \log(y_n))$ is in S, and so taking the log map of x and y makes them equivalent in our quotient space $\mathbb{R}^n/S$. Thus, this entry-wise log function maps the equivalence classes of $P/\sim$ to equivalence classes in $\mathbb{R}^n/S$. Given this connection between the spaces for which Hilbert's projective metric and the quotient l^∞ norm are defined on, we wish to find the exact relationship between Hilbert's metric and the norm-induced metric defined by

$$d_{\mathbb{R}^n/S}([x], [y]) = \|[(\log(x_1), ..., \log(x_n))] - [(\log(y_1), ..., \log(y_n))]\|_{\mathbb{R}^n/S},$$

where $x, y \in P$. If we find that the two metrics are closely related, then there is potential that results in one space could be translated into results in the other. Finding this connection leads to other potential areas of research which are discussed later. We first manipulate the quotient l^∞ norm into a form that is easier to work with:

$$\begin{aligned}
\|[x]\|_{\mathbb{R}^n/S} &= \inf_t \|x - te\|_\infty \\
&= \inf_t \max_i |x_i - t| .
\end{aligned}$$

The optimal t value which achieves the infimum will be the one that shifts all the elements of x such that $\max_i\{x_i\} = x_M$ and $\min_i\{x_i\} = x_m$ are equidistant from 0. Thus, the optimal t is given by $t = \frac{1}{2}(x_M + x_m)$, and our expression for the quotient l^∞ norm becomes

$$\|[x]\|_{\mathbb{R}^n/S} = \max_i \left| x_i - \frac{1}{2}(x_M + x_m) \right| .$$

The x_i value which will achieve the maximum will be the point x_i which is furthest away from $\frac{1}{2}(x_M + x_m)$. We know that x_M and x_m are both equidistant from $\frac{1}{2}(x_M + x_m)$, and also the points which are furthest away from $\frac{1}{2}(x_M + x_m)$. Thus, either one of $\{x_M, x_m\}$ will achieve the maximum value. Substituting x_M or x_m into our formula for the quotient l^∞ norm gives us:

$$\|[x]\|_{\mathbb{R}^n/S} = \left|x_M - \frac{1}{2}(x_M + x_m)\right| = \frac{1}{2}|x_M - x_m| = \frac{1}{2}\max_{i,j}|x_i - x_j|$$

$$\|[x]\|_{\mathbb{R}^n/S} = \left|x_m - \frac{1}{2}(x_M + x_m)\right| = \frac{1}{2}|x_M - x_m| = \frac{1}{2}\max_{i,j}|x_i - x_j|$$

Thus, we have that

$$\|[x]\|_{\mathbb{R}^n/S} = \frac{1}{2}\max_{i,j}|x_i - x_j|.$$

We are now ready to show the connection between Hilbert's projective metric and the quotient l^∞ norm. Note that $\log(x)$ where x is a vector is taken to be the entry-wise log.

$$\begin{aligned}
\|[\log(x)] - [\log(y)]\|_{\mathbb{R}^n/S} &= \frac{1}{2}\max_{i,j}|(\log(x_i) - \log(y_i)) - (\log(x_j) - \log(y_j))| \\
&= \frac{1}{2}\max_{i,j}\left|\log\left(\frac{x_i}{y_i}\right) - \log\left(\frac{x_j}{y_j}\right)\right| \\
&= \frac{1}{2}\max_{i,j}\left|\log\left(\frac{x_i y_j}{y_i x_j}\right)\right| \\
&= \frac{1}{2}\max_{i,j}\log\left(\frac{x_i y_j}{y_i x_j}\right) \\
&= \frac{1}{2}\log\max_{i,j}\left\{\frac{x_i y_j}{y_i x_j}\right\} \\
&= \frac{1}{2}d_{HPM}([x],[y]).
\end{aligned}$$

Thus, we see that Hilbert's projective metric is simply two times the metric induced by the quotient l^∞ norm applied to the entry-wise log of vectors in P.

Thompson's Metric

Let V be a real, normed vector space and let C be a positive convex cone. We define the same relation $(V, \leq_C)$ as we had for Hilbert's metric: $x \leq_C y$ if and only if $y - x \in C$. We again require $(V, \leq_C)$ to be Archimedean. We are now ready to define Thompson's metric.

Definition 3. *(Thompson, 1963) Let $x, y \in C$. We define α and β as follows:*

$$\alpha = \inf\{\lambda : x \leq_C \lambda y\} \qquad \beta = \inf\{\mu : y \leq_C \mu x\}$$

Then, $d_T(x, y) = \log(\max\{\alpha, \beta\})$ defines a metric on C.

Note that using the same α and β notation, we can write Hilbert's projective metric as

$$d_{HPM}(x, y) = \log(\alpha\beta).$$

In a previous section, we were able to show that Hilbert's projective metric is simply a scalar multiple of the metric defined using the quotient l^∞ norm. We can do something similar for Thompson's metric, but using the ordinary l^∞ norm instead. Consider $x, y \in C$. Then, we have that

$$\begin{aligned} \alpha &= \inf\{\lambda : x \leq_C \lambda y\} \\ &= \inf\{\lambda : x_i \leq \lambda y_i \ \forall\ i\} \end{aligned}$$

Taking the entry-wise log of our constraint on λ, we have

$$\begin{aligned} &\log(x_i) \leq \log(\lambda y_i) \ \forall\ i \\ \iff &\log(x_i) \leq \log(\lambda) + \log(y_i) \ \forall\ i. \end{aligned}$$

This means that $\log(x_i) - \log(y_i) \leq \log(\lambda)$ for all i. Hence, $\log(\lambda) \geq \max_i\{\log(x_i) - \log(y_i)\}$. We are trying to minimize λ, and the minimal value of λ will be α. From the previous inequality, the smallest value that $\log(\lambda)$ can be is $\max_i\{\log(x_i) - \log(y_i)\}$. Thus, we have that

$$\log(\alpha) = \max_i\{\log(x_i) - \log(y_i)\}.$$

We can do the same process for β as well.

$$\beta = \inf\{\mu : y \leq_C \mu x\} \tag{1}$$
$$= \inf\{\mu : y_i \leq \mu x_i \ \forall\ i\} \tag{2}$$

Following the same process as above, we can show that $\log(\beta) = \max_i\{\log(y_i) - \log(x_i)\}$. Putting this all together, we end up with a new way of expressing Thompson's metric:

$$\begin{aligned} d_T(x, y) &= \log(\max\{\alpha, \beta\}) \\ &= \max\{\log(\alpha), \log(\beta)\} \\ &= \max_i\{\log(x_i) - \log(y_i), \log(y_i) - \log(x_i)\} \\ &= \max_i |\log(x_i) - \log(y_i)| \\ &= \|\log(x) - \log(y)\|_\infty \end{aligned}$$

Homeomorphic Spaces

In this section, we will prove that a select few spaces related to the quotient l^∞ norm are homeomorphic.

Definition 4. *(Farenick, 2016) Let X, Y be two topological spaces. Then, a function $F : X \to Y$ is a homeomorphism if it satisfies the following properties:*

- *F is a bijection*
- *F is continuous*

- *F^{-1} is continuous*

If a homeomorphism $f : X \to Y$ exists, X and Y are said to be homeomorphic.

Homeomorphic spaces are useful to study because if two topological spaces X and Y are homeomorphic, it means they are equivalent in terms of their topological properties. For example, let $S \subseteq X$. Then, the forward image $f(S) \in Y$ will have the same topological properties as S. So the homeomorphism f preserves the properties of being open, closed, connected, compact, etc. Note that completeness is not a topological property and is not necessarily preserved by homeomorphisms. If X and Y are homeomorphic, then we can study the topological properties of X using our space Y, and vice versa.

We will use the notation $X \cong Y$ to denote that X and Y are homeomorphic. It is important to note that being homeomorphic is an equivalence relation on topological spaces, meaning it is reflexive, symmetric, and transitive. In our research, the transitive property of this equivalence relation will be useful. The transitive property is as follows: Let X, Y, Z be topological spaces. If $X \cong Y$, and $Y \cong Z$, then the transitive property of the equivalence relation tells us that $X \cong Z$.

Definition 5. *(Farenick, 2016) Let V be a vector space on a field $\mathbb{F}$ and let $\|\cdot\|_a, \|\cdot\|_b$ be two norms on V. Then, $\|\cdot\|_a$ and $\|\cdot\|_b$ are said to be equivalent if there exists $c, C > 0$ such that*

$$c\|v\|_a \leq \|v\|_b \leq C\|v\|_a \;\; \forall\, v \in V.$$

Theorem 1. *(Farenick, 2016) Let V be a finite dimensional vector space. Then, any two norms on V are equivalent.*

Theorem 2. *(Farenick, 2016) If τ and τ' are the norm topologies on a vector space V induced, respectively, by equivalent norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on V, then $\tau = \tau'$.*

Showing $(P/\sim, d_{HPM})$ Homeomorphic to $(P/\sim, d_2)$

In this section, we wish to show that the set of positive rays $P/\sim$ with Hilbert's projective metric is homeomorphic to $P/\sim$ with the "d_2" metric which we will define as follows: Let H be the surface in $\mathbb{R}^n$ given by $\{x \in \mathbb{R}^n : \prod_{i=1}^n x_i = 1\}$. For any ray $[x] \in P/\sim$, we let x_0 be the unique element of $[x]$ in H ($x_0 = [x] \cap H$). We then define our "d_2" metric on $P/\sim$ as

$$d_2([x], [y]) = \|x_0 - y_0\|_2.$$

Note that $\|\cdot\|_2$ denotes the l^2 norm. With the d_2 metric, we define the distance between rays to be the Euclidean distance between where the rays intersect the surface H. We wish to show that the positive rays ($P/\sim$) with Hilbert's projective metric is homeomorphic to $P/\sim$ with this d_2 metric we just defined.

First, we will show that $P/\sim$ with Hilbert's projective metric is homeomorphic to the quotient space $\mathbb{R}^n/S$ with the quotient l^∞ norm. To do this, we introduce a homeomorphism which will take us from $\mathbb{R}^n/S$ to the positive cone $P/\sim$. Let $F : \mathbb{R}^n/S \to P/\sim$ be defined as

$$F([(x_1, x_2, ..., x_n)]) = [(e^{x_1}, e^{x_2}, ..., e^{x_n})].$$

Let $F^{-1} : P/\sim \to \mathbb{R}^n/S$ be defined as

$$F^{-1}([(x_1, x_2, ..., x_n)]) = [(\log(x_1), \log(x_2), ..., \log(x_n))].$$

In this section, when x is a vector, e^x and $\log(x)$ will be taken to be the entry-wise exponential and log of x, respectively. First, we show that F is a bijection. Note that a function is invertible if and only if it is a bijection, so showing that F^{-1} is the inverse of F is sufficient to show that F is a bijection.

$$\begin{aligned} F^{-1}(F([x])) &= F^{-1}([e^x]) \\ &= [\log(e^x)] \\ &= [x]. \end{aligned}$$

Thus, F^{-1} is the inverse of F and so we have that F is a bijection. To show that F is a homeomorphism, we must now prove that F and F^{-1} are continuous. We want to show that for all $y \in \mathbb{R}^n/S$ and for all $\epsilon > 0$, there exists a $\delta > 0$ such that $\|[x] - [y]\|_{\mathbb{R}^n/S} < \delta \implies d_{HPM}(F([x]), F([y])) < \epsilon$.

$$\begin{aligned} d_{HPM}(F([x]), F([y])) &= 2\|\log(F([x])) - \log(F([y]))\|_{\mathbb{R}^n/S} \\ &= 2\|[\log(e^x)] - [\log(e^y)]\|_{\mathbb{R}^n/S} \\ &= 2\|[x] - [y]\|_{\mathbb{R}^n/S}. \end{aligned}$$

We can show that picking $\delta = \frac{\epsilon}{2}$ satisfies the continuity condition. The continuity of F^{-1} is also straightforward to prove, by choosing $\delta = 2\epsilon$. Thus, we have that $(P/\sim, d_{HPM}) \cong (\mathbb{R}^n/S, \|\cdot\|_{\mathbb{R}^n/S})$.

We now introduce a second norm on $\mathbb{R}^n/S$, denoted $\|\cdot\|_P$, defined as $\|[x]\|_P = \|Px\|_2$, where P is the orthogonal projection of x onto $S^\perp$. Note that if x and y are in the same equivalence class, then $Px = Py$, since we are projecting along the subspace S. Thus, the map $[x] \to Px$ is well defined. Since $\mathbb{R}^n/S$ is a finite dimensional vector space, we know that $\|\cdot\|_{\mathbb{R}^n/S}$ and $\|\cdot\|_P$ are equivalent norms. Thus, the space $\mathbb{R}^n/S$ with Hilbert's projective metric is homeomorphic to $\mathbb{R}^n/S$ with the metric induced by $\|\cdot\|_P$.

Similar to above, we now let $f : \mathbb{R}^n \to P$ be given by $f(x) = e^x$ (taken entry-wise). The inverse of f is $f^{-1} : P \to \mathbb{R}^n$ given by $f^{-1}(x) = \log(x)$ (taken entry-wise). We will show that f is a homeomorphism from $(\mathbb{R}^n, l^2)$ to (P, l^2) by showing that f and its inverse are continuous. To prove f is continuous, we must show that for all $y \in \mathbb{R}^n$ and for all $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x - y\|_2 < \delta \implies \|e^x - e^y\|_2 < \epsilon$ for all $x \in \mathbb{R}^n$. First note that the function $h : \mathbb{R} \to \mathbb{R}_+$ given by $h(x_i) = e^{x_i}$ is continuous, which means for all $\frac{\epsilon}{\sqrt{n}} > 0$, we can find a $\delta > 0$ such that $|e^{x_i} - e^{y_i}| < \frac{\epsilon}{\sqrt{n}}$ whenever $|x_i - y_i| < \delta$. Picking this value of delta, we have

$$
\begin{aligned}
\|x-y\|_2<\delta &\iff \sqrt{(x_1-y_1)^2+...+(x_n-y_n)^2}<\delta \\
&\implies \sqrt{(x_i-y_i)^2}<\delta \;\forall\, i \\
&\implies |x_i-y_i|<\delta \;\forall\, i \\
&\implies |e^{x_i}-e^{y_i}|<\frac{\epsilon}{\sqrt{n}} \;\forall\, i \\
&\implies \sqrt{(e^{x_1}-e^{y_1})^2+...+(e^{x_n}-e^{y_n})^2)}<\sqrt{\left(\frac{\epsilon}{\sqrt{n}}\right)^2+...+\left(\frac{\epsilon}{\sqrt{n}}\right)^2} \\
&\implies \|e^x-e^y\|_2<\sqrt{n}\left|\frac{\epsilon}{\sqrt{n}}\right| \\
&\implies \|e^x-e^y\|_2<\epsilon
\end{aligned}
$$

Thus, the exponential map is continuous. Now we must check the inverse function, which is the entry-wise log map. Following the same process,

$$
\begin{aligned}
\|x-y\|_2<\delta &\implies |x_i-y_i|<\delta \;\forall\, i \\
&\implies |\log(x_i)-\log(y_i)|<\frac{\epsilon}{\sqrt{n}} \;\forall\, i \\
&\implies \|\log(x)=\log(y)\|_2<\epsilon.
\end{aligned}
$$

We see that the inverse function is also continuous. Altogether, we have that f is a homeomorphism from $(\mathbb{R}^n, l^2)$ to (P, l^2). We introduce the following theorem which will be useful to connect this to the d_2 metric defined on $P/\sim$.

Theorem 3. *If $f: X \to Y$ is a homeomorphism, and $A \subseteq X$, then f is a homeomorphism from A to $f(A)$.*

We have that $\|[x]\|_P$ is defined as $\|Px\|_2$, where Px is the projection of x onto $S^\perp$. Note that $S^\perp \subseteq \mathbb{R}^n$. We have just shown that $f: \mathbb{R}^n \to P$ is a homeomorphism, so by theorem 3, we have that f is a homeomorphism from $S^\perp$ to $f(S^\perp)$. Note that the entry-wise exponential function maps $S^\perp$ to the surface H, which means that $f(S^\perp) = H$. This is because $S^\perp$ is the set of all vectors whose elements sum to 0, so when you exponentiate each element of the vectors and then multiply them, you get $e^{x_1+x_2+...+x_n} = e^0 = 1$. So we have that $(S^\perp, l^2)$ is homeomorphic to (H, l^2).

To connect this to our metric space $(P/\sim, d_2)$, recall that in this space, we represent each positive ray by the intersection of the ray with H, and we define the distance between the rays as the l^2 distance between where they intersect H. Using the function $h: P/\sim \to H$ given by $h([x]) = \frac{x}{(\prod_i x_i)^{\frac{1}{n}}}$, it is straightforward to show that $(P/\sim, d_2)$ and (H, l^2) are homeomorphic.

We now have everything we need to show that our space $(P/\sim, d_{HPM})$ is homeomorphic to $(P/\sim, d_2)$. Using F, we showed that $(P/\sim, d_{HPM}) \cong (\mathbb{R}^n/S, \|\cdot\|_{\mathbb{R}^n/S})$. The quotient l^∞

norm is equivalent to the norm $\|[x]\|_P = \|Px\|_2$ on $\mathbb{R}^n/S$, so $(\mathbb{R}^n/S, d_{HPM})$ is homeomorphic to $(\mathbb{R}^n/S, \|\cdot\|_P)$. The metric induced by the $\|\cdot\|_P$ norm simply takes the l^2 distance between points which lie on $S^\perp$, so it is homeomorphic to $(S^\perp, l^2)$. Using f, we showed $(S^\perp, l^2)$ is homeomorphic to (H, l^2), where H is the set $\{x \in \mathbb{R}^n : \prod_i x_i = 1\}$. The d_2 metric we defined on $P/\sim$ represents each positive ray as the unique point where the ray intersects H, and takes the l^2 distance between these points on H. Thus $(P/\sim, d_2)$ is homeomorphic to (H, l^2). Putting this together, we know the following spaces are homeomorphic:

$$(P/\sim, d_{HPM}) \cong (\mathbb{R}^n/S, \|\cdot\|_{\mathbb{R}^n/S}) \cong (\mathbb{R}^n/S, \|\cdot\|_P) \cong (S^\perp, l^2) \cong (H, l^2) \cong (P/\sim, d_2)$$

Since being homeomorphic is transitive, $(P/\sim, d_{HPM}) \cong (P/\sim, d_2)$.

Showing (P, d_T) Homeomorphic to (P, l^2)

Using a similar argument, we can also show that the positive cone P with Thompson's metric is homeomorphic to P with the ordinary l^2 metric. First we show that (P, d_T) is homeomorphic to $(\mathbb{R}^n, l^\infty)$. Again, we will let $f(x) = e^x$ and $f^{-1}(x) = \log(x)$. We will show that $f : \mathbb{R}^n \to P$ is continuous. To do this, we prove that for all $y \in \mathbb{R}^n$ and for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|x - y\|_\infty < \delta$, then $d_T(f(x) - f(y)) < \epsilon$. Note that

$$\begin{aligned} d_T(f(x) - f(y)) &= d_T(e^x - e^y) \\ &= \|\log(e^x) - \log(e^y)\|_\infty \\ &= \|x - y\|_\infty. \end{aligned}$$

So picking $\delta = \epsilon$ ensures that $\|x - y\|_\infty < \delta \implies d_T(f(x) - f(y)) < \epsilon$, and we have that f is continuous. It is also straightforward to show that f^{-1} is continuous by again picking $\delta = \epsilon$. Altogether, we see that (P, d_T) is homeomorphic to $(\mathbb{R}^n, l^\infty)$.

Now, we show that $(\mathbb{R}^n, l^\infty)$ is homeomorphic to (P, l^2). Again, we will use our bijective function f, and prove the continuity of f and f^{-1}. To prove the continuity of f, we will show that for all $y \in \mathbb{R}^n$ and for all $\epsilon > 0$, we can find a $\delta > 0$ such that $\|x - y\|_\infty < \delta$ implies that $\|f(x) - f(y)\|_2 < \epsilon$. Let n be the length of our vectors x and y. Again, we will use the fact that $h : \mathbb{R} \to \mathbb{R}_+$ given by $h(x_i) = e^{x_i}$ is continuous, which means for all $\frac{\epsilon}{\sqrt{n}} > 0$, we can find a $\delta > 0$ such that $|x_i - y_i| < \delta_i$ implies $|e^{x_i} - e^{y_i}| < \frac{\epsilon}{\sqrt{n}}$.

We can show that picking $\delta_M = \max\{\delta_i\},\ 1 \le i \le n$ works to prove the continuity of f. With this choice of delta, we have that $\|x - y\|_\infty = \max_i |x_i - y_i| < \delta_M$. This means that $|x_i - y_i| < \delta_M$ for all $1 \le i \le n$. By the continuity of the exponential function, this means that $|e^{x_i} - e^{y_i}| < \frac{\epsilon}{\sqrt{n}}$ for all i. Thus,

$$\begin{aligned}
\|F^{-1}(x) - F^{-1}(y)\|_2 &= \|e^x - e^y\|_2 \\
&= \sqrt{(e^{x_1} - e^{y_1})^2 + ... + (e^{x_1} - e^{y_1})^2} \\
&< \sqrt{\left(\frac{\epsilon}{\sqrt{n}}\right)^2 + ... + \left(\frac{\epsilon}{\sqrt{n}}\right)^2} \\
&= \sqrt{n\left(\frac{\epsilon}{\sqrt{n}}\right)^2} \\
&= \sqrt{n}\left|\frac{\epsilon}{\sqrt{n}}\right| \\
&= \epsilon.
\end{aligned}$$

Thus, f continuous.

Now we prove the continuity of f^{-1}. We want to show that for all $y \in P$ and for all $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x - y\|_2 < \delta$ implies $\|\log(x) - \log(y)\|_\infty < \epsilon$, for all $x \in P$. We know that $\log(x_i)$ is a continuous function for $x_i \in \mathbb{R}_+$. Thus, we know that for any $\epsilon > 0$, there exists a $\delta_i > 0,\ 1 \leq i \leq n$ such that $|x_i - y_i| < \delta_i \implies |\log(x_i) - \log(y_i)| < \epsilon$. Let $\delta_m = \min\{\delta_i\}$. We make the assertion that $\|x - y\|_2 < \delta_m$ and show that this choice of delta works to prove the continuity of f^{-1}.

$$\begin{aligned}
\|x - y\|_2 < \delta_m &\iff \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2} < \delta_m \\
&\iff \sum_{i=1}^{n}(x_i - y_i)^2 < \delta_m^2 \\
&\implies (x_i - y_i)^2 < \delta_m^2 \qquad \forall\ i,\ 1 \leq i \leq n \\
&\implies |x_i - y_i| < \delta_m \qquad \forall\ i,\ 1 \leq i \leq n.
\end{aligned}$$

By the continuity of the log function, we then have that $|\log(x_i) - \log(y_i)| < \epsilon$ for all i, $1 \leq i \leq n$. Thus, $\max_i |\log(x_i) - \log(y_i)| < \epsilon$ and so we have that $\|\log(x) - \log(y)\|_\infty < \epsilon$. Thus, f^{-1} is also continuous, and we have that $(\mathbb{R}^n, l^\infty)$ is homeomorphic to (P, l^2). Altogether, we have proven the following homeomorphisms:

$$(P, d_T) \cong (\mathbb{R}^n, l^\infty) \cong (P, l^2).$$

Since being homeomorphic is transitive, we have that $(P, d_T) \cong (P, l^2)$.

Characterization of Compact Sets in Hilbert's Projective Metric

We have shown that Hilbert's projective metric on the positive cone is homeomorphic to the Euclidean topology on the positive cone intersected with the surface $H = \prod_{i=1}^{n} x_i = 1$. In Kohlberg (1982), the author provides a relatively simple proof of the Perron-Frobenius theorem, without the restriction that the mapping A must satisfy additivity. Necessary to this proof is showing that the cone C which is contained in the positive orthant is compact under Hilbert's projective metric. We propose a theorem which can simplify the proof of this. Due to our previous results, we know that C is compact under Hilbert's projective metric if $C \cap H$ is a compact set under the standard Euclidean topology. To do this, we first need a definition:

Definition 6. *(Seeger, 1999) Let X be a Hilbert space and $C \subseteq X$ be a cone. Then, a nonempty set $S \subseteq X$ is said to be a shell for the cone C if $0 \notin C$, and $C = \{\alpha p : \alpha \in \mathbb{R}_+, p \in S\}$.*

Theorem 4. *Let $C \subseteq \mathbb{R}^n_+$ be a cone contained in the positive orthant. Let $U = \{x \in \mathbb{R}^n : \|x\| = 1\}$ and let $H = \{x \in \mathbb{R}^n : \prod_i x_i = 1\}$. Then, the following four statements are equivalent:*

1. *$H \cap C$ is compact under the ordinary Euclidean topology.*
2. *$U \cap C$ is compact under the ordinary Euclidean topology.*
3. *C admits a compact shell.*
4. *C is compact under Hilbert's projective metric.*

Proof. ($2 \iff 1$)
Let C be a cone contained in the positive orthant of $\mathbb{R}^n$. Let H be the surface given by $\prod_{i=1}^n x_i = 1$. Finally let U be the boundary of the unit ball in $\mathbb{R}^n$: $U := \{x \in \mathbb{R}^n : \|x\| = 1\}$.

We define a function $f : U \to H$ as follows:

$$f(x) = \frac{x}{(\prod_{i=1}^n x_i)^{\frac{1}{n}}}$$

Note that f is continuous and maps $U \cap C$ to the portion of H which intersects the cone, $H \cap C$. Our function f also has a continuous inverse $f^{-1} : H \to U$ which can be defined as follows:

$$f^{-1}(x) = \frac{x}{\|x\|}.$$

Note that f maps $x \in U$ to the unique multiple of x which lies on H. Further, f^{-1} maps $x \in H$ to its unique multiple which lies on U. Thus, f and f^{-1} are inverses, and we have that f is bijective. We see that f is a homeomorphism from U to H. Therefore, $U \cap C$ is compact if and only if $H \cap C$ is compact

($1 \iff 4$)
We have shown earlier that the cone C with Hilbert's projective metric is homeomorphic to $C \cap H$ with the usual Euclidean metric. Thus, if $H \cap C$ is compact, then it must be that C is compact under Hilbert's projective metric, and vice versa.

($2 \implies 3$)

Note that $U \cap C$ is a shell of C. Thus, if $U \cap C$ is compact, C admits a compact shell.

$(3 \implies 2)$
Let S be a compact shell of C. Consider the function $f : S \to U \cap C$ given by $f(p) = \frac{p}{\|p\|}$. This is a surjective function and is also continuous since $p \neq 0$. So f takes an element of S and maps it to its unique position on $U \cap C$. If S is compact, the forward image $f(S) = U \cap C$ must also be compact.

□

Corollary 1. *Let $C \subseteq \mathbb{R}^n_+$ be a closed cone contained in the positive orthant. Then, C is compact in Hilbert's projective metric.*

Proof. Again, let $U = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Using the previous theorem, we know that C is compact in Hilbert's projective metric if and only if $U \cap C$ is compact. Thus, we can prove the compactness of C by showing that $U \cap C$ is compact in the ordinary Euclidean norm. We can now use the Heine-Borel theorem which states that any subset of a Euclidean space is compact if and only if it is closed and bounded. The set U is the set of all elements in $\mathbb{R}^n$ with unit norm, thus we know that U is bounded and so $U \cap C$ is also bounded. We also know that U is a closed as a subset of $\mathbb{R}^n$, so if C is a closed cone, then $U \cap C$ is also closed in $\mathbb{R}^n$. We have that $U \cap C$ is closed and bounded, and is therefore compact. Therefore, C must be compact in Hilbert's projective metric. □

Geometry of Quotient l^∞ Norm Unit Ball

One way to better understand the quotient l^∞ norm is to study the geometry of its unit ball. That is, studying what the set $\{x \in \mathbb{R}^n/S : \|x\|_{\mathbb{R}^n/S} \leq 1\}$ looks like geometrically. We will first introduce some terminology which will help explain the geometry of the unit ball.

Definition 7. *(Farenick, 2016) Let V be a vector space and let W be a subset of V. Then, we say W is a convex set if $tu + (1-t)v \in W$ for all $u, v \in W$, and for all $t \in [0, 1]$.*

Definition 8. *(Farenick, 2016) Let V be a vector space and let W be a convex subset of V. Then, an element $v \in W$ is an extreme point of W if v is not interior to any line segment contained in W (i.e., if there exists $v_1, v_2 \in W$ and $t \in (0, 1)$ such that $v = tv_1 + (1-t)v_2$, then $v = v_1 = v_2$).*

The orthogonal complement of our set $S = \text{span}\{e = (1, 1, ..., 1)\}$ will be useful in studying the geometry of the quotient norm unit ball. First let's cover what the orthogonal complement ($S^\perp$) is in this case.

$$\begin{aligned} S^\perp &= \{x \in \mathbb{R}^n : \langle x, s \rangle = 0 \ \ \forall\, s \in S\} \\ &= \{x \in \mathbb{R}^n : \langle x, (1, 1, ..., 1) \rangle = 0\} \\ &= \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0\} \end{aligned}$$

We see that $S^\perp$ is the set of all vectors whose elements sum to 0. To better understand the quotient l^∞ unit ball, we first consider the ordinary l^∞ unit ball on $\mathbb{R}^n$. The l^∞ unit ball in n

dimensions is the n-dimensional hypercube centered about the origin. When we quotient out our subspace S, what we are essentially doing is taking the points in $\mathbb{R}^n$, and projecting them along the direction of S onto the hyperplane $S^{\perp}$. Note that the n-dimensional hypercube is convex, and the projection of a convex set in $\mathbb{R}^n$ onto a subspace is itself convex. Also note that any extreme point of the quotient norm unit ball will be a projection of an extreme point of the original unit ball, however, not necessarily all points of the original unit ball will be projected onto extreme points, because it is possible they could be projected to the interior of the resulting set. The shape of the quotient l^{∞} unit ball can be found by taking the extreme points of the ordinary l^{∞} unit ball, projecting them onto $S^{\perp}$ along the direction of S, and then taking the convex hull of the resulting points. We will call the resulting set K. The question becomes, when we take these extreme points and project them onto $S^{\perp}$ and take the convex hull of the points, which points will remain as extreme points of the resulting set K, and which points will get mapped to the interior of K?

Let's first consider the 2 dimensional quotient l^{∞} unit ball. The ordinary l^{∞} unit ball looks like a square centred about the origin. Our subspace S is a line that runs through the points $(-1,-1)$ and $(1,1)$. To find the quotient l^{∞} unit ball in 2 dimensions, we take the points in the square and project them onto $S^{\perp}$, as shown below:

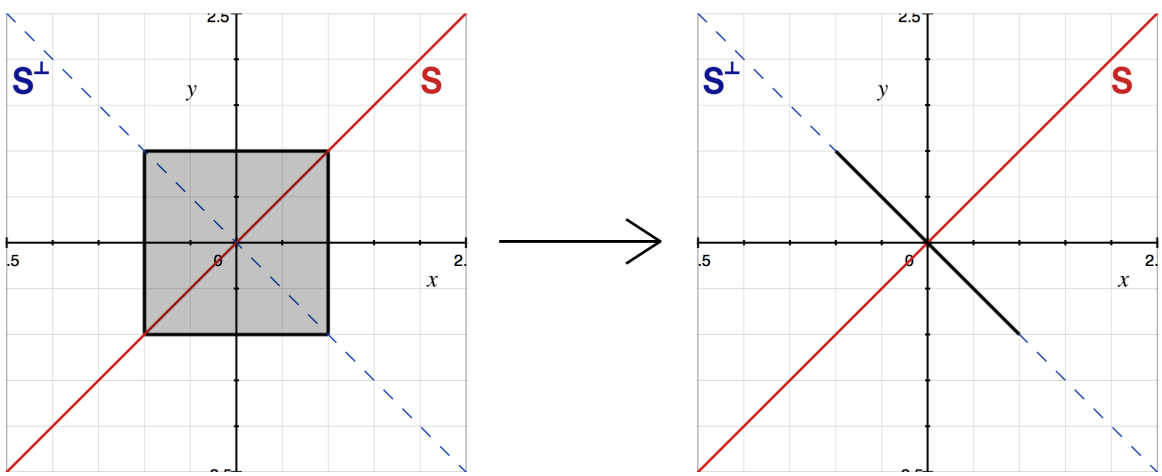

Figure 1: On the left hand side, we have the l^{∞} unit ball given in grey. The right hand side depicts the resulting set when the l^{∞} unit ball is projected onto $S^{\perp}$.

We see that the quotient l^{∞} unit ball in two dimensions is simply a line, which has 2 extreme points. Now we consider the quotient l^{∞} unit ball in 3 dimensions. To do this, we again first look at the normal l^{∞} unit ball in 3D, which is a cube centered about the origin.

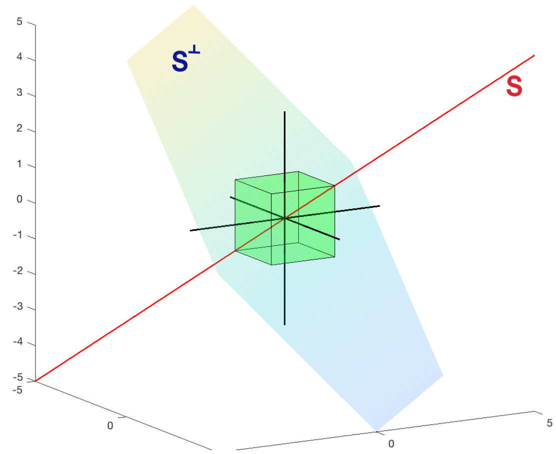

Figure 2: The l^∞ unit ball in $\mathbb{R}^3$, plotted with the subspace $S = \text{span}\{e\}$ as well as the plane $S^\perp$, which is the set of all vectors who sum to 0.

To find the shape of the l^∞ unit ball, we find the orthogonal projection of the cube onto the plane $S^\perp$. A way to visualize this is to rotate the cube so we are looking directly downwards on our subspace S and imagine a light is shining down on the cube and casting a shadow onto $S^\perp$. The shape of this shadow is the shape of the quotient l^∞ norm.

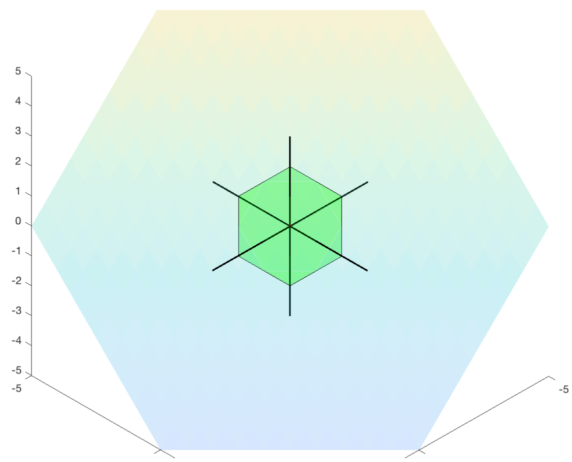

Figure 3: The l^∞ unit ball as in figure 2, rotated such that we are looking directly downwards at the line S.

From figure 3, we see that the resulting shape we will get is the regular hexagon (in green). Thus, we see that the quotient l^∞ unit ball in 3 dimensions is the regular hexagon, which has 6 extreme points.

In the 2 and 3 dimensional cases, it is easy to visualize what the quotient l^∞ unit ball would look like, but we lose the ability to effectively visualize it when we consider higher dimensions. Thus, we wish to find a way to calculate what the extreme points of the unit ball are. Following a similar procedure, we first consider the extreme points of the regular l^∞ unit ball in $\mathbb{R}^n$. These extreme points are $(\pm 1, \pm 1, ..., \pm 1)$.

There are two extreme points of the l^∞ unit ball for which we can tell immediately that they will be projected into the interior of K. Consider the points $(1, 1, ..., 1)$ and $(-1, -1, ..., -1)$. Both of these points lie on the subspace S and thus, when we project them down in the direction of S, these points will get mapped to the origin. Thus, they will be mapped to the interior of the projection and will not be extreme points of K. We will now study which of the remaining extreme points get mapped to extreme points of the projection.

Let $v \in \mathbb{R}^n$ be an extreme point of the l^∞ which has at least one $+1$ and one -1 entry. We wish to take v and quotient out S by projecting it onto $S^\perp$. Let x be the projection of v onto $S^\perp$, which is given by

$$x = v - \frac{\langle v, e \rangle}{\langle e, e \rangle} e$$

Recall that $e = (1, 1, ..., 1)$. How can we check if this point x will be an extreme point of the resulting set? To do this, we will use the following theorem:

Theorem 5. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a linear function, and let K be a compact, convex subset of $\mathbb{R}^n$. Then, g achieves its maximum (and minimum) values at an extreme point of K.*

This theorem is widely used in linear programming, and is the reason why the simplex algorithm only needs to check the extreme points of a compact, convex set to find the minimum or maximum. Note that our resulting set K obtained when we project the hypercube onto $S^\perp$ will be convex and compact. Thus, if we can find a linear function which achieves its maximum on K at the point x, and for which no other distinct point $y \in K$ achieves the same maximum, we then we know that x is an extreme point of K. The question becomes, what choice of function g should we make to check this?

In the 4-dimensional case, all extreme points of the l^∞ unit ball with at least one $+1$ and one -1 will be permutations of the points $(1, 1, 1, -1)$, $(1, 1, -1, -1)$, and $(1, -1, -1, -1)$. When these points get projected down onto $S^\perp$, they will become permutations of the points $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-3}{2})$, $(1, 1, -1, -1)$, and $(\frac{3}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2})$, respectively. Consider the projection $x^* = (\frac{3}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2})$. We want to find a linear function g for which $g(x^*) > g(y)$, where $y \neq x$ is any other extreme point of K. Picking the function $g(x) = x_1 - x_2 - x_3 - x_4$, we see that $g(x^*) = 3$, and it can be shown that $3 > g(y)$, when y is any other distinct extreme point of K. What if we now considered the point $x^* = (1, 1, -1, -1)$? Then, choosing $g(x) = x_1 + x_2 - x_3 - x_4$, we can also show that x^* is the maximum point of g on our set K, and no other point in K achieves the same maximum.

In fact, following this pattern and checking each point, we can show that every extreme point v of the 4-dimensional hypercube which has at least one $+1$ and one -1 entry gets projected onto an extreme point of K. The way we do this is as follows: fix v to be an extreme point of the hypercube which doesn't lie on S, and let $g(x) = \langle x, v \rangle$. Let x^* be the projection of v onto $S^\perp$. Then, the function g is maximized on K at the unique point x^*, which means that x^* is an extreme point of K. We now explore if this result will generalize to n dimensions.

Theorem 6. *Let $S = span\{(e = (1, 1, ..., 1)\}$. Then, the unit ball of the quotient l^∞ norm given by $\|x\|_{\mathbb{R}^n/S} = \inf_t \|x - te\|_\infty$ has $2^n - 2$ extreme points.*

Proof. The ordinary l^∞ unit ball in $\mathbb{R}^n$ is an n-dimensional hypercube with 2^n extreme points. As mentioned above, the two extreme points of the form $(1, 1, ..., 1)$ and $(-1, -1, ..., -1)$ get mapped to the interior of K when we quotient out S, leaving us with $2^n - 2$ extreme points to consider.

Fix v to be an extreme point of the l^∞ unit ball which doesn't lie on S. Let $g(x) = \langle x, v\rangle$. We will let $x^* \in K$ denote the projection of v onto $S^\perp$. To show that all $2^n - 2$ of these extreme points get mapped to extreme points of K, we will show that g achieves its maximum on K at the unique point x^*. Recall that $x^* = v - \frac{\langle v,e\rangle}{\langle e,e\rangle}e$. Then,

$$\begin{aligned} g(x^*) &= \langle x^*, v\rangle \\ &= \langle v, v\rangle - \frac{\langle v, e\rangle}{\langle e, e\rangle}\langle e, v\rangle. \end{aligned}$$

We want to show that $g(x^*)$ is greater than g evaluated at any other point in K. We know that g will be maximized at an extreme point of K, so we will only consider potential extreme points of K. Note that only extreme points of the hypercube can be mapped to extreme points of K, so we will let $w \neq v$ be another extreme point of the hypercube with at least one $+1$ and one -1 entry. Then, the projection of w is given by $y^* = w - \frac{\langle w,e\rangle}{\langle e,e\rangle}e$. Then,

$$g(y^*) = \langle w, v\rangle - \frac{\langle w, e\rangle}{\langle e, e\rangle}\langle e, v\rangle.$$

If we can show that $g(x^*) > g(y^*)$, then we have shown that all projections of extreme points of the hypercube with at least one $+1$ and one -1 entry are also extreme points of K, the unit ball of the quotient l^∞ norm. Showing that $\langle e, e\rangle\langle v-w, v\rangle > \langle e, v\rangle\langle v-w, e\rangle$ is equivalent to showing $g(x^*) > g(y^*)$, as shown below.

$$\begin{aligned} &\langle e, e\rangle\langle v - w, v\rangle > \langle e, v\rangle\langle v - w, e\rangle \\ \iff &\langle e, e\rangle(\langle v, v\rangle - \langle w, v\rangle) > \langle e, v\rangle(\langle v, e\rangle - \langle w, e\rangle) \\ \iff &\langle v, v\rangle\langle e, e\rangle - \langle v, e\rangle\langle e, v\rangle > \langle w, v\rangle\langle e, e\rangle - \langle w, e\rangle\langle e, v\rangle \\ \iff &\langle v, v\rangle - \frac{\langle v, e\rangle}{\langle e, e\rangle}\langle e, v\rangle > \langle w, v\rangle - \frac{\langle w, e\rangle}{\langle e, e\rangle}\langle e, v\rangle \\ \iff &g(x^*) > g(y^*) \end{aligned}$$

Can we show that this is true? First, note that $\langle e, e\rangle = n$, where n is the dimension of the vectors in our space. Now note that $\langle v, e\rangle$ is equal to the number of $+1$ entries of v minus the number of -1 entries of v. Thus, $-(n-2) \leq \langle v, e\rangle \leq n-2$, and we have that $\langle e, e\rangle > |\langle e, v\rangle|$.

Now let's take a look at $\langle v - w, e\rangle$. This is simply the sum of the elements of the vector $v - w$. So all of the indices of v and w which are the same contribute 0 to the inner product.

All of the indices for which $v_i = 1$ and $w_i = -1$ contribute 2 to the inner product. Similarly, all of the indices for which $v_i = -1$ and $w_i = 1$ contribute -2 to the inner product.

Let k_2 be the number of indices for which $v_i = 1$ and $w_i = -1$. This is because at these indices, $(v - w)_i = 2$. Let k_{-2} be the number of indices for which $v_i = -1$ and $w_i = 1$. This is because at these indices, $(v - w)_i = -2$. Finally, let k_0 be the number of indices for which $v_i = w_i$. At these indices, $(v - w)_i = 0$. Now let's take a look at $\langle v - w, e\rangle$. This is simply the sum of the elements of the vector $v - w$. Thus,

$$\begin{aligned}\langle v - w, e\rangle &= 2k_2 - 2k_{-2} + 0k_0 \\ &= 2k_2 - 2k_{-2}.\end{aligned}$$

Now let's look at $\langle v - w, v\rangle$. This is similar to the previous inner product, except now we are taking the inner product with v, which has both $+1$ and -1 entries. Whenever $v_i = -1$ and $w_i = 1$, the $-2k_{-2}$ term in the expression for $\langle v - w, e\rangle$ will be multiplied by -1. The rest will stay the same. Putting this together, we have

$$\begin{aligned}\langle v - w, v\rangle &= 2k_2 - 2k_{-2}(-1) + 0k_0 \\ &= 2k_2 + 2k_{-2}.\end{aligned}$$

We made the assumption that $v \neq w$, which means that one of $\{k_2, k_{-2}\}$ must be positive, and so we know that $\langle v - w, v\rangle$ is a positive quantity. Note that $-(2k_2 + 2k_{-2}) \leq 2k_2 - 2k_{-2} \leq 2k_2 + 2k_{-2}$ and thus we have that $|2k_2 - 2k_{-2}| \leq 2k_2 + 2k_{-2}$, which is the same as saying $\langle v - w, v\rangle \geq |\langle v - w, e\rangle|$.

We now have everything we need to prove the inequality. We start with $\langle e, e\rangle > |\langle e, v\rangle|$. Note that $\langle v - w, v\rangle$ is a positive quantity, so multiplying by it does not change the inequality. This gives us

$$\langle e, e\rangle\langle v - w, v\rangle > |\langle e, v\rangle|\,\langle v - w, v\rangle.$$

Now, we apply the fact that $\langle v - w, v\rangle \geq |\langle v - w, e\rangle|$, which tells us that

$$\begin{aligned}\langle e, e\rangle\langle v - w, v\rangle &> |\langle e, v\rangle|\,\langle v - w, v\rangle \\ &\geq |\langle e, v\rangle|\,|\langle v - w, e\rangle| \\ &= |\langle e, v\rangle\langle v - w, e\rangle|\end{aligned}$$

Altogether, we have that

$$\begin{aligned}&\langle e, e\rangle\langle v - w, v\rangle > |\langle e, v\rangle\langle v - w, e\rangle| \\ \Longrightarrow\ &\langle e, e\rangle\langle v - w, v\rangle > \langle e, v\rangle\langle v - w, e\rangle \\ \Longleftrightarrow\ &g(x^*) > g(y^*).\end{aligned}$$

Thus, we have shown that x^* is an extreme point of the quotient l^∞ unit ball, and so the quotient l^∞ unit ball must have $2^n - 2$ extreme points.

□

With this result, we are able to study the geometry of the quotient l^∞ norm unit ball in higher dimensions. We know that all extreme points other than $\pm(1, 1, ..., 1)$ on the l^∞ unit ball will become extreme points of the projection, so we can find the extreme points of the quotient l^∞ norm by projecting these $2^n - 2$ extreme points onto $S^\perp$. Taking the convex hull of these extreme points gives us our resulting quotient l^∞ unit ball.

Consider the quotient l^∞ unit ball in 4 dimensions. We know that when we take the 4 dimensional hypercube and quotient out the 1 dimensional subspace S, we will be left with a 3 dimensional unit ball. By projecting extreme points of the 4D hypercube onto $S^\perp$, we can find the extreme points of the quotient l^∞ norm unit ball in 4D coordinates. To study what the 3 dimensional shape of the unit ball looks like, we want to express these points in 3 dimensions. To do this, we can use the orthonormal basis for $\mathbb{R}^4/S$ given by

$$\left\{ v_1 = \begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{bmatrix}, v_2 = \begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{bmatrix}, v_3 = \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \\ 0.5 \end{bmatrix} \right\}.$$

Note that each vector sums to 0, and thus lies in $S^\perp$. We can construct a matrix A by letting row i of our matrix be given by vector v_i. It can be shown that the matrix A maps v_1, v_2, v_3 to $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, respectively. So we have that A is an orthogonal mapping from $\mathbb{R}^n/S$ to $\mathbb{R}^3$. Let v be an extreme point of the l^∞ unit ball in 4 dimensions. Then, Av is the coordinates of the corresponding extreme point of the quotient l^∞ unit ball in 3 dimensions. Taking the convex hull of all $2^4 - 2 = 14$ extreme points, we can show that the quotient l^∞ unit ball in 4 dimensions is the rhombic dodecahedron.

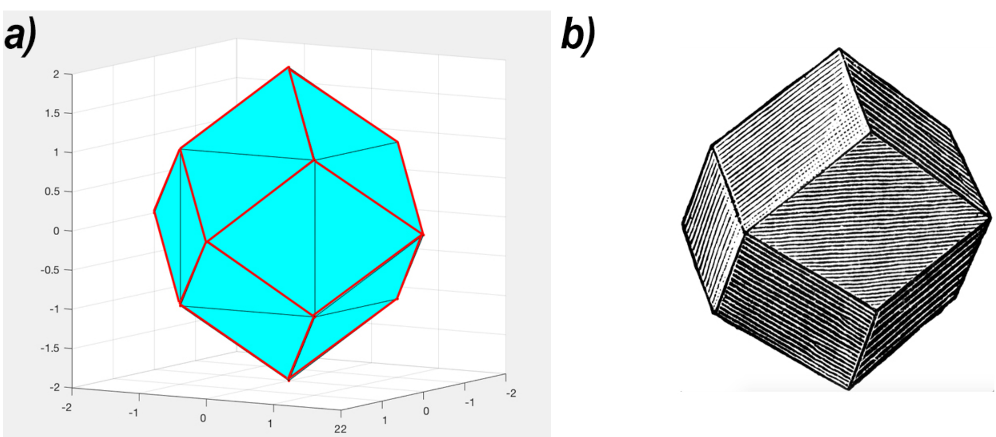

Figure 4: a) Rhombic dodecahedron generated by taking the convex hull of the extreme points the quotient l^∞ unit ball. Faces are outlined in red. Image generated using MATLAB. b) Illustration of a rhombic dodecahedron (Chambers and Chambers, 1881).

A similar process can be done to find the extreme points of the unit ball in higher dimensions. However, these shapes do not seem to have common names nor are they widely studied. Being able to calculate the extreme points of the quotient l^∞ ball can allow us to find a simpler formula for the quotient l^∞ operator norm, which is an area for further research.

The Perron-Frobenius Theorem

In this section, we will demonstrate that our results allow for a simplification of the proof of the main result in *The Perron-Frobenius theorem without additivity* written by Elon Kohlberg (1981). To do so, we must first introduce a definition and state the Perron-Frobenius theorem.

Definition 9. *(Kohlberg, 1982) Let A be a non-negative $n \times n$ real matrix. Then, A is said to be primitive if $A^l > 0$ for some integer $l > 0$.*

In the following theorem, when x and y are vectors, we say that x is greater than (or equal to) y if x_i is greater than (or equal to) y_i for all i. The Perron-Frobenius theorem can be stated as follows:

Theorem 7. *(Kohlberg, 1982) Let A be a continuous mapping $A : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ which satisfies the following conditions:*

- *$A(x + y) = Ax + Ay$ (additivity)*
- *$A(\lambda x) = \lambda Ax \; \forall \; \lambda \geq 0$ (homogeneity of degree 1)*
- *For some integer $l > 0$, $x \geq y \implies A^l x > A^l y$ (primitivity).*

When these conditions are met, the Perron-Frobenius theorem states that

1. *There exists an eigenvector $x_0 > 0$, unique up to a scalar multiple, such that $Ax_0 = \lambda_0 x_0$ for some $\lambda_0 > 0$.*
2. *$\frac{A^n x}{\|A^n x\|} \to \frac{x_0}{\|x_0\|}$ for all $x \geq 0$, which is to say that $A^n x$ converges in direction to x_0.*

The paper by Kohlberg (1982) actually proves a nonlinear generalization of this theorem. Other nonlinear extensions of the Perron-Frobenius theorem have been given prior to his work, but they only applied to result (1) in the theorem. The main result Kohlberg's paper is that if the additivity condition on A is dropped, both results (1) and (2) still hold.

The proof of this result involves the use of a lemma which only applies to compact metric spaces (C, d). In Kohlberg's proof, C is a cone contained in the positive orthant $\mathbb{R}^n_+$, and the metric d they are working with is Hilbert's projective metric. In order to use the lemma, they must show that (C, d) is compact. Using corollary 1 in our paper, this becomes much simpler. All they must show is that C is closed, and they have the result that (C, d) is a compact metric space. In fact, in Kohlberg's paper, C is a closed set, so using our result, the compactness condition is clearly satisfied. This demonstrates one application for which our characterization of Hilbert's projective metric allows for easier characterization of topological properties of sets under Hilbert's projective metric.

Why Hilbert's Projective Metric?

As stated above, Hilbert's projective metric is useful for proving the Perron-Frobenius theorem. To be more specific, the proof of the Perron-Frobenius theorem developed by Garret

Birkhoff relies on the fact that linear transformations given by positive matrices are contraction mappings when applied to the nonnegative orthant (Kohlberg and Pratt, 1982). One thing we wished to investigate when we started our research of Hilbert's projective metric and the quotient norm was to look into if we could find a different metric under which positive matrices would become contraction mappings. In particular, if we developed a metric which was closely related to Hilbert's projective metric using the quotient norm, we wanted to investigate if this metric could also be used in the proof of the Perron-Frobenius theorem. However, what we found was that Hilbert's projective metric and the metric we defined using the quotient l^∞ norm were nearly equivalent, with one simply being a scalar multiple of one another. So this metric we developed using the quotient space $\mathbb{R}^n/S$ was not "new", and would not change the proof of the Perron-Frobenius theorem.

The paper *The contraction mapping approach to the Perron-Frobenius theory: why Hilbert's metric?* by Kohlberg and Pratt (1982) essentially shows that finding a different metric which isn't Hilbert's projective metric to prove the Perron-Frobenius theorem is not possible, as stated in the main result of their paper:

Theorem 8. *(Kohlberg and Pratt, 1982) Let $\geq$ be the ordering induced by a closed, pointed cone K in $\mathbb{R}^n$, and let d_{HPM} be Hilbert's projective metric on K. Then, any positive linear transformation is a contraction with respect to d_{HPM}. Conversely, if D is a projective metric on K such that every positive linear transformation is a contraction with respect to D, then there exists a continuous strictly increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $D(x,y) = f(d_{HPM}(x,y))$ for all $x, y > 0$.*

What this theorem is saying, is that up to a transformation by a continuous strictly increasing function f, Hilbert's projective metric is the only metric for which every positive linear transformation is a contraction. So it is impossible to metric which is not a transformation of Hilbert's projective metric by a function f as defined, and for which all positive linear transformations are contractive. Knowing this allowed us to recognize that it would not be possible to find a different metric to prove the Perron-Frobenius theorem, and so our research should be shifted onto other applications of the quotient l^∞ norm.

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